# Syllabus of "Computaional Complexity 2022/2023" 

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## 1 Bibliography

The main textbook of the course is

- [AB] Arora, Barak. Computational Complexity: A Modern Approach. Cambridge University Press, 2007.

Additional material on communication complexity and proof complexity are in

- [J] Jukna. Boolean Function Complexity. Springer, 2012.
- [RY] Rao, Yehudayoff. Communication Complexity and Applications. Cambridge University Press, 2020.

A proof from the last part of the course is only published, as far as I know, in

- $[\mathrm{K}]$ Knuth, the Art of Computer Programming, Vol 4, fascicle 6, Page 55-56.

All reference to a textbook refer to book $[\mathrm{AB}]$ unless specified otherwise.

## 2 Introduction

- Introduction
- Chapter 0
- Parity on $n$ variables requires CNF formulas of size $\Theta\left(n 2^{n}\right)$
- Equality of two strings of $n$ bits requires $n$ bits of communication


## 3 Turing machines, Universality, Uncomputability, the class $P$

- Sections 1.1, 1.2, 1.3, 1.4
- Sections 1.5, 1.6
- Exercises 1.2, 1.3, 1.5, 1.8, 1.14
- Read and ponder the statement of Exercise 1.6, 1.7 and 1.9


## 4 NP and NP-completeness

- Karp reductions
- NP-completeness
- Non deterministic Turing Machines
- Cook-Levin theorem proved via Circuit-SAT (see Section 6.1)
- Decision vs Seach
- coNP, EXP, NEXP
- Implications of P vs NP
- Sections 2.1, 2.2, 2.3 (Section 2.3.4 is optional), 2.4, 2.5, 2.6, 2.7
- Section 6.1
- Exercises 2.1, 2.2, 2.4, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.14, 2.15, 2.16, 2.17, 2.18, 2.21, 2.23, 2.24, 2.25, 2.26, 2.27, 2.29, 2.30, 2.31, 2.34.


## 5 Time Hierarchy Theorems, Oracles

- Hierarchy theorem for deterministic time
- Hierarchy theorem for non-deterministic time (just the statement)
- Ladner theorem (just the statement)
- Turing Machines with oracles
- P vs NP does not relativize w.r.t. to Oracles
- Sections 3.1, 3.2, 3.3, 3.4


## 6 Space complexity, NL-completeness

- The Graph of configurations
- PSPACE, TQBF is PSPACE-complete
- Savitch's Theorem
- Lospace reductions and NL-completeness
- $\mathrm{NL}=\mathrm{coNL}$
- Sections 4.1, 4.2, 4.3
- Exercises 4.1, 4.2, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9, 4.10, 4.11


## 7 Randomized computation, RP, coRP, BPP, ZPP

- Appendix A. 2
- Sections 7.1, 7.2 (no 7.2.4), 7.3, 7.4, 7.5 (no 7.5.3)
- Exercises 7.1, 7.4, 7.5, 7.6


## 8 Interactive Proofs and PSPACE

- randomness in interaction is necessary
- public vs private randomness
- IP, AM, MA classes
- protocol for Graph Non Isomorphism
- coNP $\in$ IP
- $\mathrm{IP}=\mathrm{PSPACE}$
- Sections 8.1, 8.2, 8.3
- Sections 8.2.2 and 8.2.3 are optional, but the statement of Theorems 8.12 and 8.13 are part of the program
- Exercises 8.1, 8.2, 8.6, 8.8


## 9 Circuits and Polynomial Hierarchy

- BPP vs P/poly
- BPP in the second level of the hierarchy
- Karp-Lipton theorem
- if GI is NP-complete, then PH collapses to second level
- Circuit size complexity
- Theorem 1.29 on Jukna's book [J]
- Sections 5.1, 5.2, 6.1, 6.2, 6.4, 6.5, 6.6, 7.5, 8.2.4
- Exercises 5.1, 6.3, 6.5, 6.6


## 10 Decision trees

- decision trees
- worst case depth complexity
- certificate complexity
- just the statement that $\mathrm{bs}(\mathrm{f})<=2(\mathrm{~s}(\mathrm{f}))^{4}$ [Huang'19]
- Section 12.1, 12.2, 12.3, 12.4
- Exercises 12.1, 12.2, 12.3, 12.4, 12.5, 12.6, 12.7


## 11 Communication complexity

- A deterministic protocol induces partition in monocromatic combinatorial rectagles
- Methods of lower bound: fooling set, tiling, rank
- randomized communication complexity
- public vs private coins
- worst case vs average case (Minimax theorem)
- Sections 13.1, 13.2.1, 13.2.2, 13.2.3
- Exercises 13.1, 13.3, 13.5, 13.9, 13.1013 .19
- Chapter 1 up to page 15 included. [RY]
- Chapter 3 [RY]


## 12 Proof complexity

- proof systems
- p-simulation
- resolution
- tree-like resolution $==$ decision trees
- prover-delayer game for tree-like resolution [K]
- ordeding principle is hard for tree like resolution $[\mathrm{K}]$
- ordering principle is easy for resolution
- pigeonhole principle is hard for resolution
- size-width relation
- Tseitin formulas
- Section 15.1, 15.2.1
- Section 18.1, 18,4, 18.5, 18.6, 18,7 [J]
- Theorem 18.17 just the statement [J]
- Exercises: weakening is not necessary in resolution refutations
- Exercises: in tree-like resolution we can avoid resolving on the same variable along any proof path.

